Direct Calculation of PDFs in Lattice QCD

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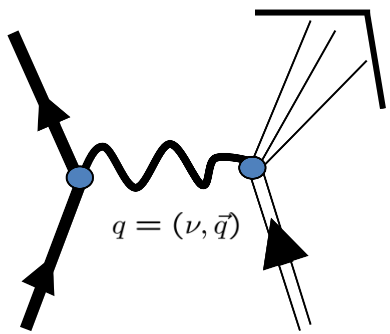
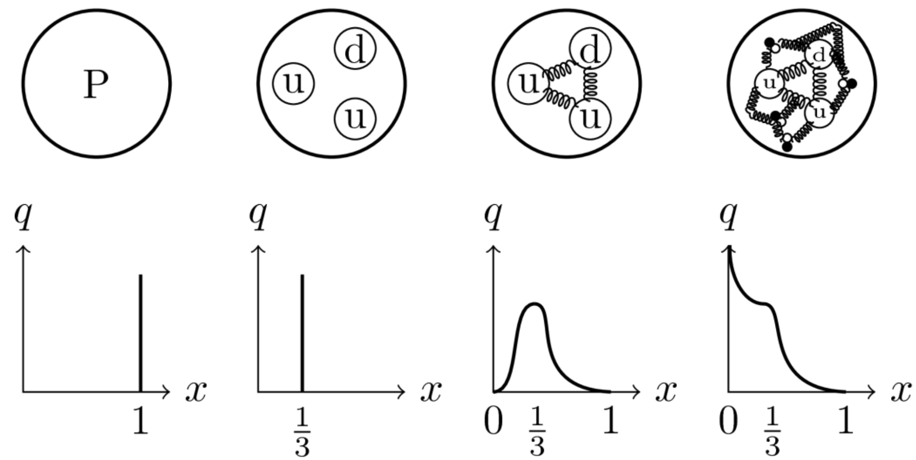
Outline

- Proton structure and quark distributions
- Quark quasi-distributions
- Extracting quark distributions from the quasi-distributions
- Lattice computation of the matrix elements
- Unpolarized, helicity and transversity distributions
- The case of free quarks
- Momentum smearing and PDFs at large nucleon momentum
- Summary and outline

The proton structure





Elastic | 3 free quarks | 3 bound quarks | Quarks + Gluons

Cross sections

↓

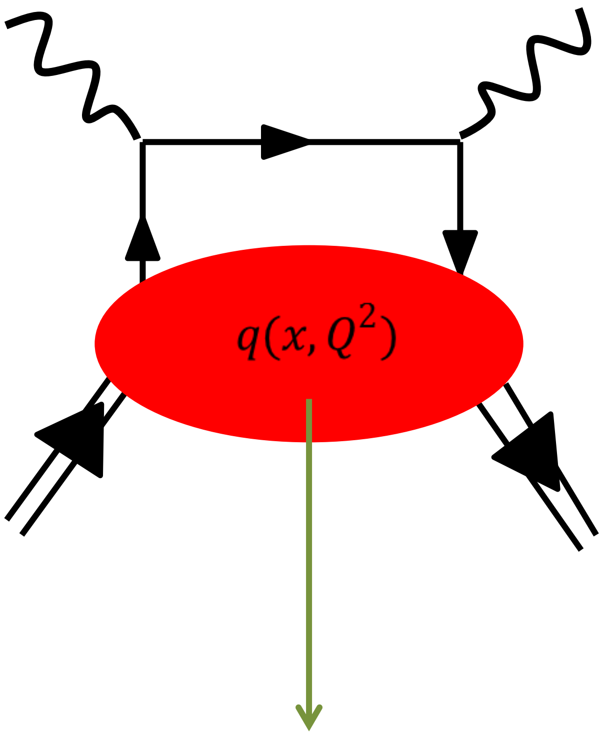
Structure Functions

↓

Quark and Gluon Distributions

In the Bjorken limit

$Q^2, \nu \to \infty, \qquad x = \frac{Q^2}{2P \cdot q}$



Parton distributions

QCD + OPE

$$\int_0^1 dx x^{n-2} F_2(x, Q^2) = \sum_i a_n^{(i)} C_n^{(i)}(Q^2)$$

$$\langle P | \mathcal{O}_{\mu_1 \cdots \mu_n} | P \rangle = a_n P_{\mu_1} \cdots P_{\mu_2}$$

Moments of the parton distributions

The x dependence of the distributions

Inverse Mellin transform $a_n = \int dx\, x^{n-1} q(x)$ $\quad q(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dn\, x^{-n} a_n$

Taking $\mu_1 = \mu_2 = \cdots = \mu_n = +$

$$\boxed{q(x) = \int_{-\infty}^{+\infty} \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P | \bar{\psi}(\xi^-) \gamma^+ W(\xi^-, 0) \psi(0) | P \rangle}$$

$W(\xi^-, 0) = e^{-ig \int_0^{\xi^-} A^+(\eta^-) d\eta^-}$ (Wilson line)

- Light cone correlations in the nucleon rest frame
- Equivalent to the distributions in the Infinite Momentum Frame
- Light cone dominated $\xi^2 = t^2 - z^2 \sim 0$
- Not calculable on Euclidian lattice $t^2 + z^2 \sim 0$

Quasi Distributions

X. Ji, "Parton Physics on a Euclidean Lattice," PRL 110 (2013) 262002.

Matrix elements $\langle P|O^{\mu_1\mu_2\cdots\mu_n}|P\rangle = 2a_n^{(0)}\Pi^{\mu_1\mu_2\cdots\mu_n}$ $\quad P = (P_0, 0, 0, P_3)$

Setting $\mu_1 = \mu_2 = \cdots = \mu_{2k} = 3$

$$\boxed{\langle P|O^{3\cdots 3}|P\rangle = 2\tilde{a}_{2k}^{(0)}(P_3)^{2k}\sum_{j=0}^{k}\mu^j\frac{(2k-j)!}{j!\,(2k-2j)!} \equiv 2\tilde{a}_{2k}(P_3)^{2k}}$$

With $\boxed{\mu = M^2/4(P_3)^2}$

Defining: $\tilde{a}_n(P_3) = \int_{-\infty}^{+\infty} dx\, x^{n-1} \tilde{q}(x, P_3)$

Mellin transformation implies in $\boxed{\tilde{q}(x, P_3) = \int_{-\infty}^{+\infty} \frac{dz}{4\pi} e^{-izk_3} \langle P | \bar{\psi}(z) \gamma^3 W(z,0) \psi(0) | P \rangle}$

$k_3 = xP_3$ (Parton momentum)

$W(z) = e^{-ig \int_0^z A_3(z')dz'}$ (Wilson line)

- Nucleon moving with finite momentum in the z direction
- Pure spatial correlation
- Can be simulated on a lattice

What are these quasi-distributions? Do they have a partonic interpretation?

The light cone distributions:

$$x = \frac{k^+}{P^+}$$

$$0 \leq x \leq 1$$

Distributions can be defined in an Infinite Momentum Frame: P_3, P^+ goes to infinite

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Quasi distributions:

P_3 large but finite

Usual partonic interpretation is lost

$x < 0$ or $x > 1$ is possible

But they can be related to each other!

Extracting quark distributions from quark quasi-distributions

Infrared region untouched when going from
a finite to an infinite momentum

Infinite momentum frame: $P_3 \to \infty, \Lambda\ fixed$

$$\boxed{q(x,\mu) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi} Z_F(\mu)\right\} + \frac{\alpha_s}{2\pi}\int_x^1 q^{(1)}\left(\frac{x}{y},\mu\right) q_{bare}(y)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)}$$

Finite momentum: $\Lambda \to \infty, P_3\ fixed$

$$\boxed{\tilde{q}(x,\Lambda,P_3) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi}\widetilde{Z_F}(\Lambda,P_3)\right\} + \frac{\alpha_s}{2\pi}\int_{x/x_c}^1 \tilde{q}^{(1)}\left(\frac{x}{y},\Lambda,P_3\right) q_{bare}(y)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)}$$

$x_c \sim \Lambda/P_3$ Largest value at which the calculations are meaningful

Solving for the quark distributions

$$\boxed{\begin{aligned} q(x,\mu) = {} & \tilde{q}(x,\Lambda,P_3) - \frac{\alpha_s}{2\pi}\tilde{q}(x,\Lambda,P_3)\delta Z_F^{(1)}\left(\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right) \\ & - \frac{\alpha_s}{2\pi}\int_{-1}^{1} Z^{(1)}\left(\frac{x}{y},\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right)\tilde{q}(x,\Lambda,P_3)\frac{dy}{|y|} + \mathcal{O}(\alpha_s^2) \end{aligned}}$$

$$\delta Z_F^{(1)} = \widetilde{Z}_F - Z_F$$

$$Z^{(1)} = \tilde{q}^{(1)} - q^{(1)}$$

Desired quantity

From the lattice

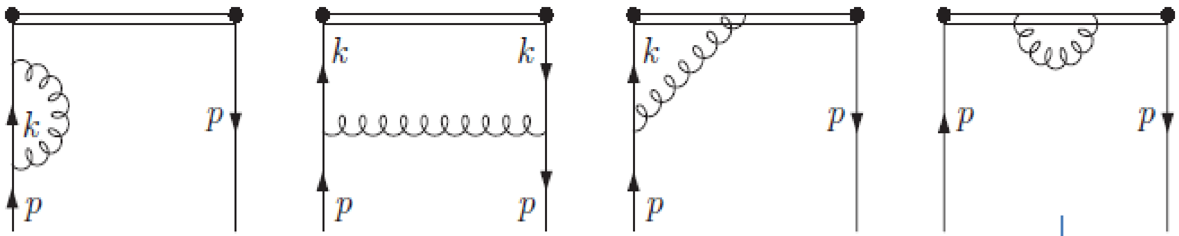
From pQCD

X. Xiong, X. Ji, J.-H. Zhang and Y. Zhao,
“One loop matching for parton distributions:Nonsinglet case,”PRD90 (2014) 014051.

C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, k. Hadjiyiannakou, K. Jansen, FS and C. Wiese,
“A Lattice Calculation of Parton Distributions,” PRD92 (2015) 014502.

Perturbative QCD

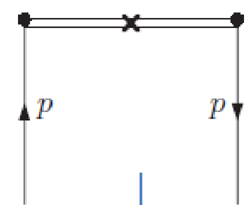
Wilson line



$$\tilde{q}(x,\Lambda,P_3)=\int_{-\infty}^{+\infty}\frac{dz}{4\pi}e^{izk_3-\delta m|z|}\langle P|\bar{\psi}(z)\gamma^3 W(z,0)\psi(0)|P\rangle$$

Linear divergence comes from this diagram

This extra term removes the linear divergence



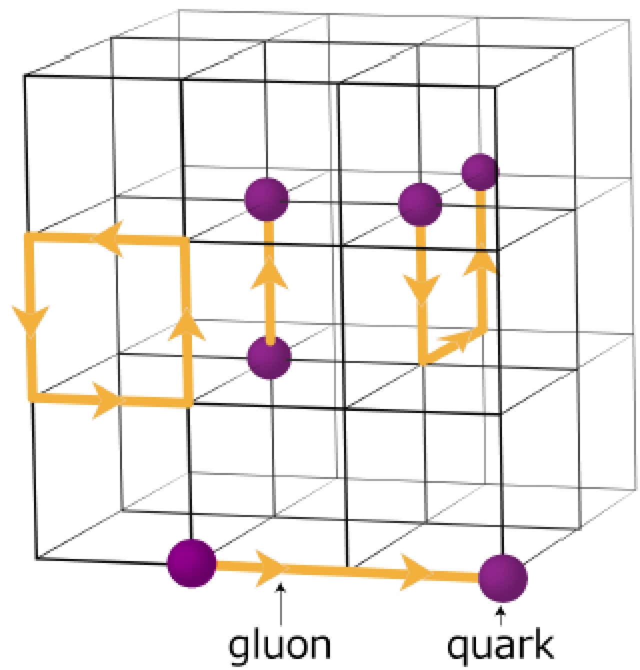
Mass counterterm introduced to remove the linear div.

J. W. Chen, X. Ji and J. H. Zhang,
``Improved quasi parton distribution through Wilson line renormalization,''
arXiv:1609.08102.

T. Ishikawa, Y. Q. Ma, J. W. Qiu and S. Yoshida,
``Practical quasi parton distribution functions,''
arXiv:1609.02018.

Lattice QCD

- We introduce a 4D hypercubic lattice:
  - ⋆ quark fields on lattice sites,
  - ⋆ gluon fields on lattice links.
- Gauge invariant objects:
  - ⋆ Wilson loops,
  - ⋆ quarks and antiquarks connected with a gauge link.
- Lattice as a regulator:
  - ⋆ UV cut-off – inverse lat. spac. a^{-1},
  - ⋆ IR cut-off – inverse lat. size L^{-1}.
- Remove the regulator:
  - ⋆ continuum limit $a \to 0$,
  - ⋆ infinite volume limit $L \to \infty$.



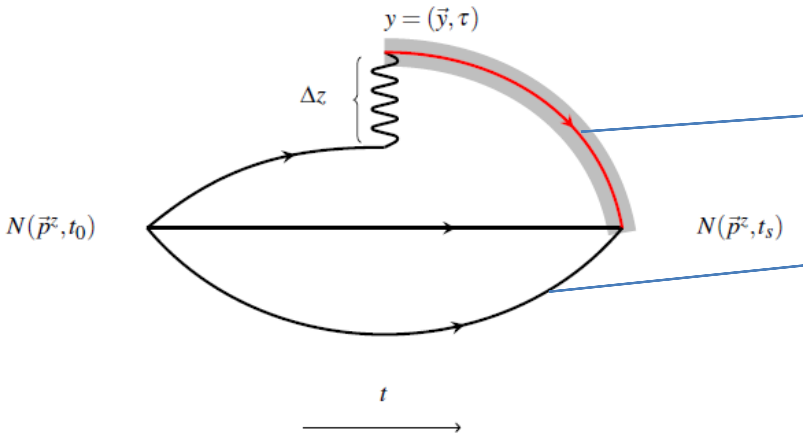
Source: JICFuS, Tsukuba

We want: $$h(P_3, z) = \langle P | \bar{\psi}(z)\gamma_3 W_3(z,0)\psi(0) | P \rangle$$

Let : $$C^{3pt}(t,\tau,0) = \langle N_\alpha(\vec{P},t)\mathcal{O}(\tau)\overline{N_\alpha}(\vec{P},0)\rangle$$

$$N_\alpha(\vec{P},t) = \Gamma_{\alpha\beta}\sum_{\vec{x}} e^{i\vec{P}\cdot\vec{x}}\epsilon^{abc}u^a_\beta(x)\left(d^{b^T}(x)\mathcal{C}\gamma_5 u^c(x)\right)$$

$$\mathcal{O}(z,\tau,Q^2=0) = \sum_{\vec{y}} \bar{\psi}(y+z)\gamma_3 W_3(y+z,y)\psi(y)$$



All to all propagators needed

Stochastic source method is used

Point source method is used

Flavour structure: u - d

Extraction of the matrix elements

We use maximally twisted mass fermions

$$\frac{C^{3pt}(t,\tau,0;P_3)}{C^{2pt}(t,0;P_3)} = \frac{-iP_3}{E} h(P_3,z), \quad 0 \ll \tau \ll t$$

$8a, 10a$ Source – sink separation

$32^3 \times 64$ Lattice

$\beta = \frac{6}{g_0^2} = 1.95$ $a \approx 0.082\, fm \quad N_f = 2+1+1$

Maximally twisted mass ensemble: $a\mu = 0.0055 \Longrightarrow m_{ps} \cong 370$ MeV

$$P_3 = \frac{2\pi}{L}, \frac{4\pi}{L}, \cdots$$

Configurations

1000 gauge configurations
15 point source forward propagators
02 Stochastic propagators

30000 measurements

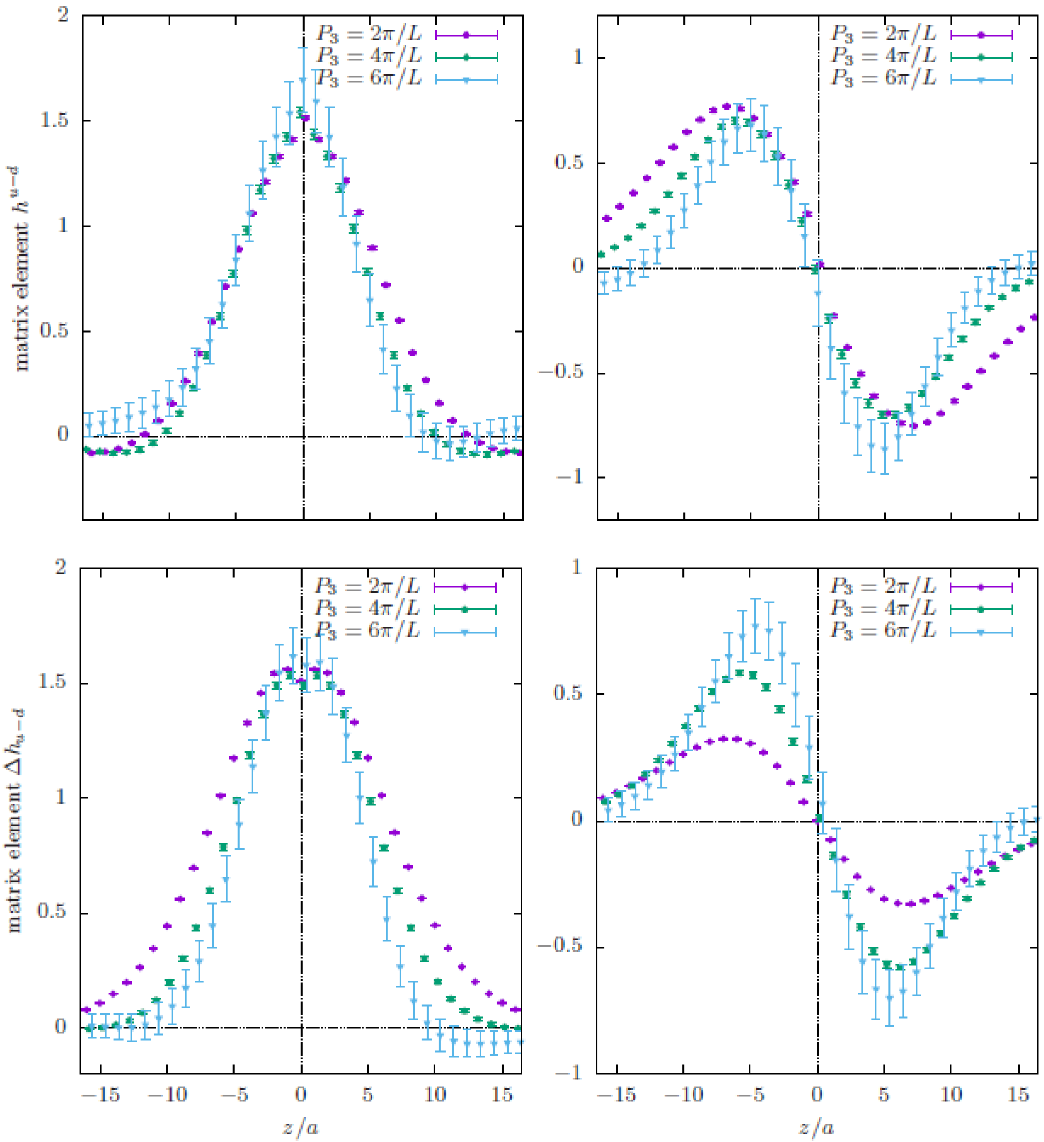
Operators

Unpolarized γ_3

Helicity $\gamma_3\gamma_5$

Transversity $\gamma_3\gamma_j$

Matrix Elements



Unpolarized

5 steps of HYP Smearing

Polarized

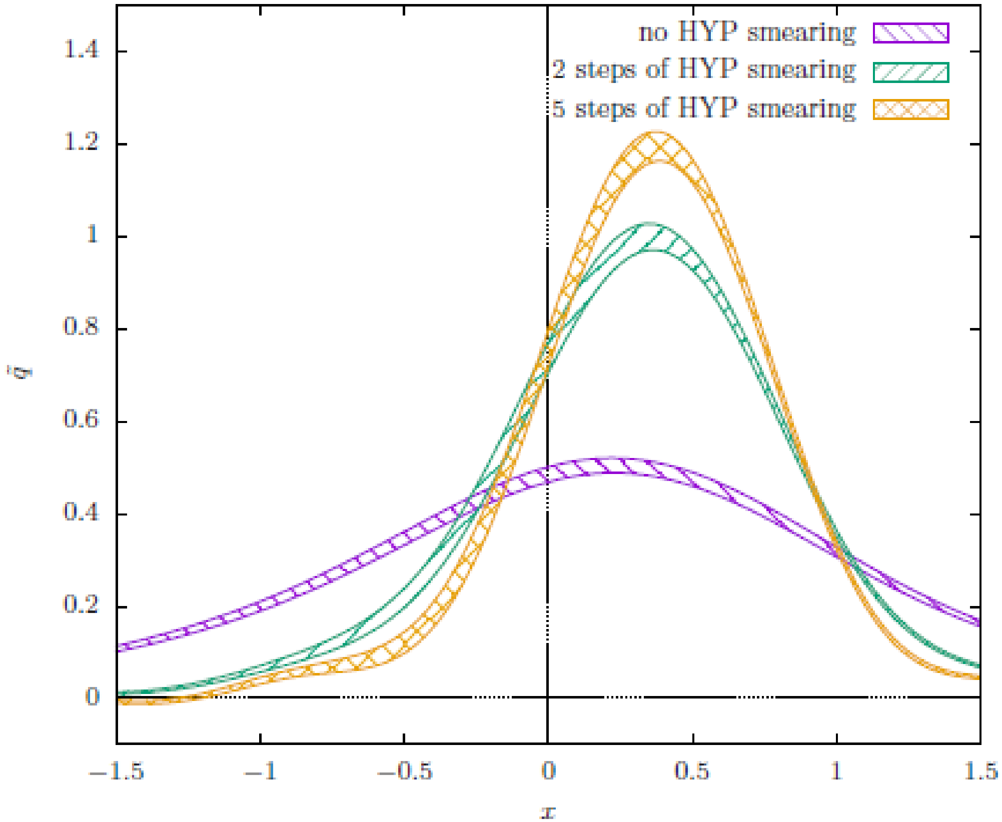
Real part

Imaginary part

HYP Smearing

It replaces a given gauge link with some average over neighbouring links, *i.e*. ones from the hypercubes attached to it

Crude substitute for renormalization



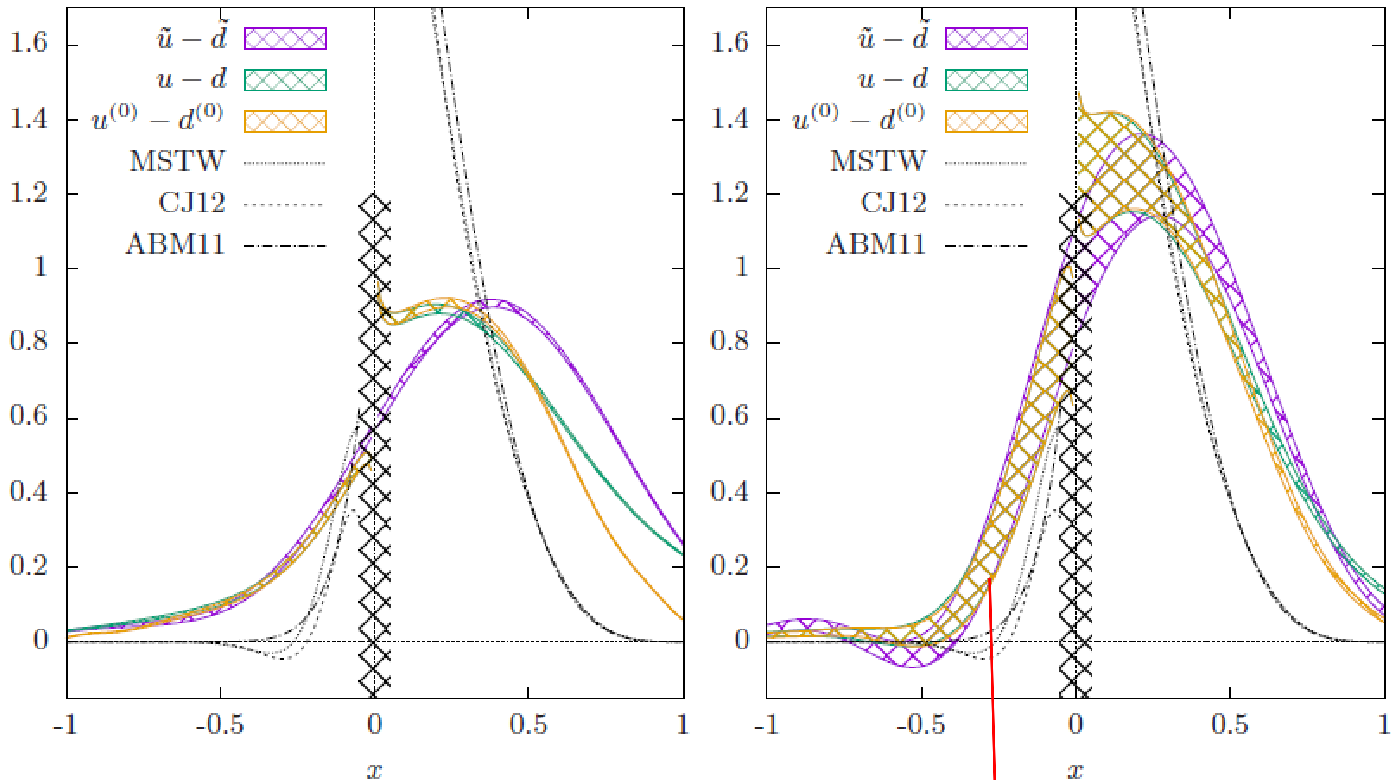
$$P_3 = \frac{4\pi}{L}$$

Parameters

$$\alpha_s = \frac{6}{4\pi\beta} \approx 0.245$$

$$\Lambda = \frac{1}{a} \cong 2.5\ \text{GeV}$$

$u(x) - d(x)$



$P_3 = \frac{4\pi}{L}$ (0.98 GeV)

$P_3 = \frac{6\pi}{L}$ (1.47 GeV)

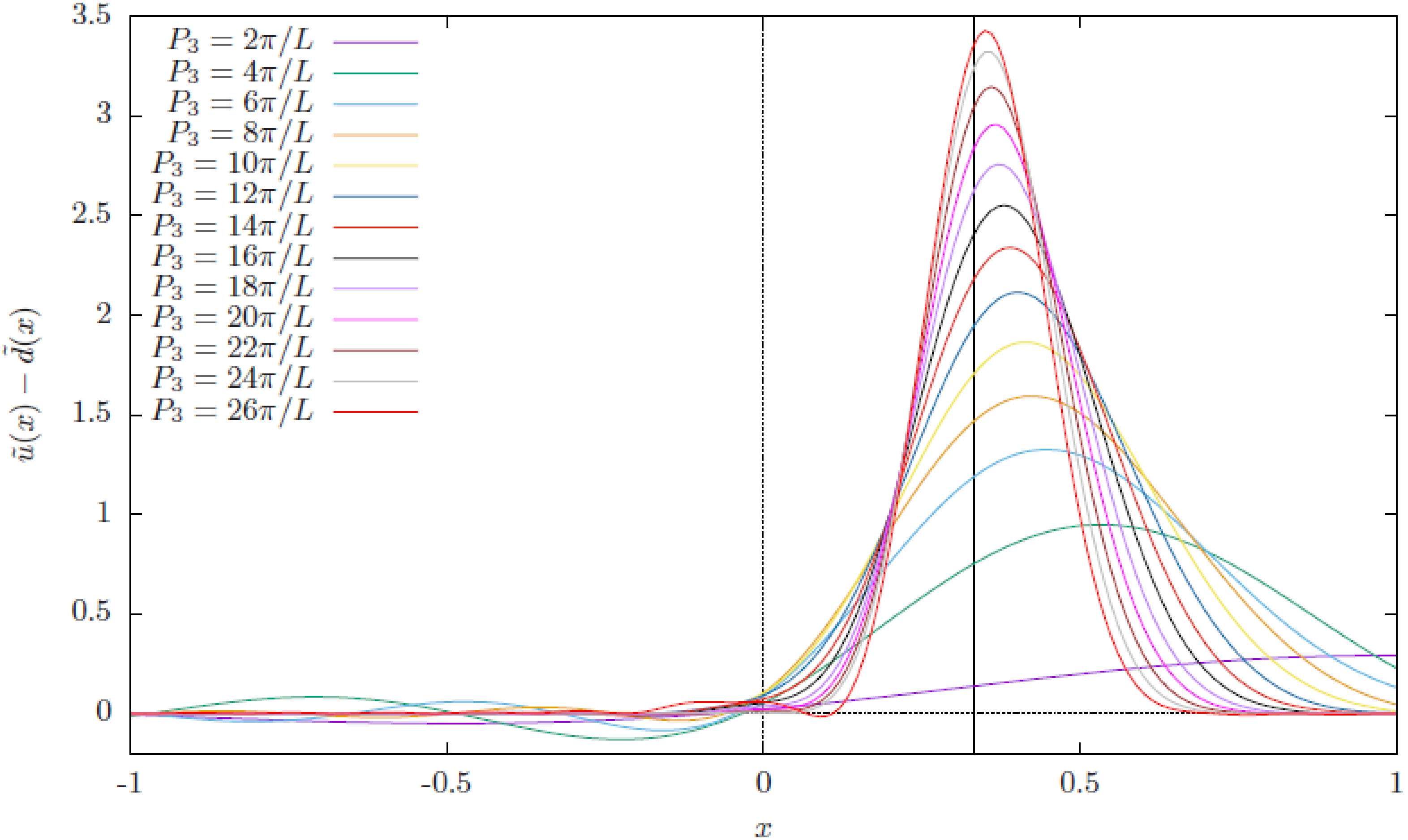
Crossing relation:

$$\bar{q}(x) = -q(-x)$$

$$\int_0^1 dx\left(\bar{d}(x) - \bar{u}(x)\right) = 0.187 \pm 0.055$$

NMC: $\int_0^1 dx\left(\bar{d}(x) - \bar{u}(x)\right) = 0.148 \pm 0.039$

The case of free quarks.



Tends to a Dirac delta at 1/3, as expected

Momentum smearing

- We would like to study the PDFs at larger momenta
  Problem: poor signal

- Possible solution by Bali *et al.* in arXiv:1602.05525

- Alter Gaussian smearing so that in momentum space the desired momentum is modeled

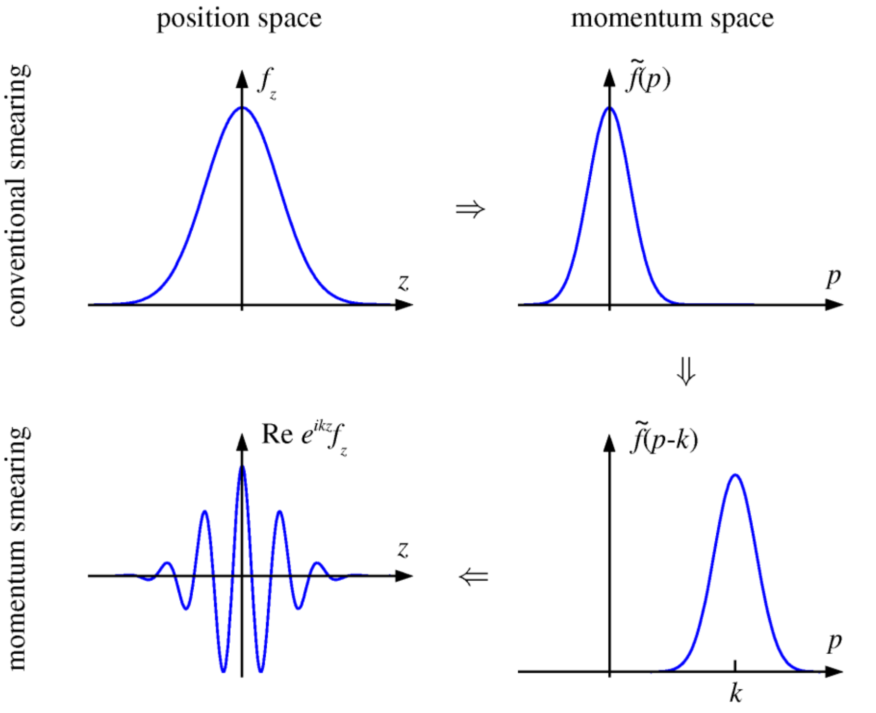
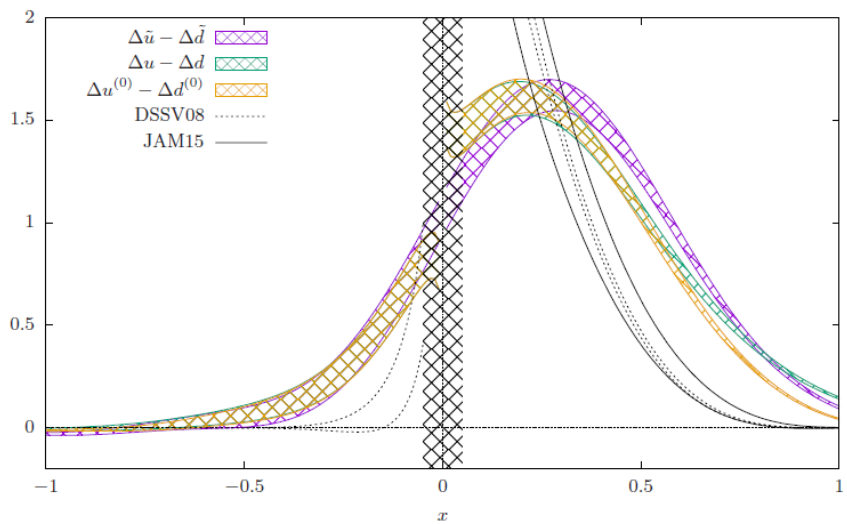


Figure from arXiv:1602.05525

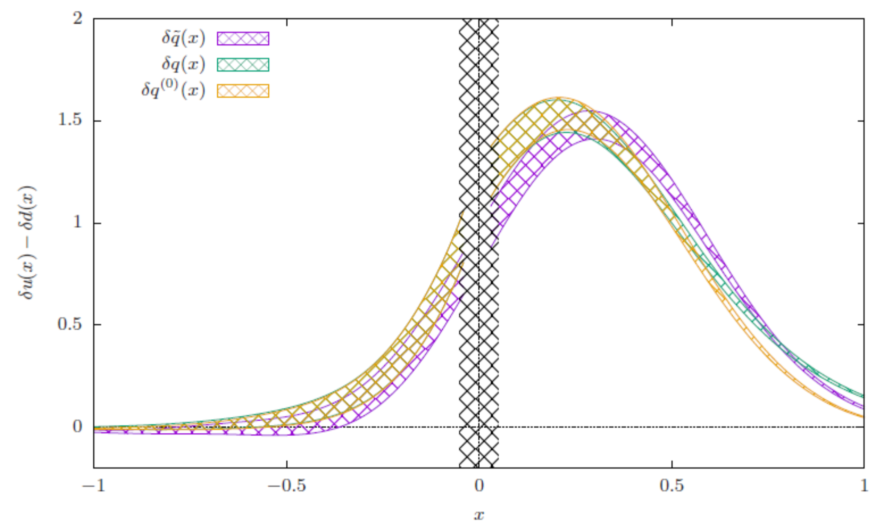
$$S_M(k)\psi(x) = \frac{1}{1+8\kappa}\left[\psi(x) + \kappa\sum e^{ik\hat{j}}U_j(x)\psi(x+\hat{j})\right]$$

Results for $P_3 = \frac{6\pi}{L}$

C. Alexandrou *et al.*,
``New Lattice Results for Parton Distributions,"
arXiv:1610.03689



Helicity



Transversity

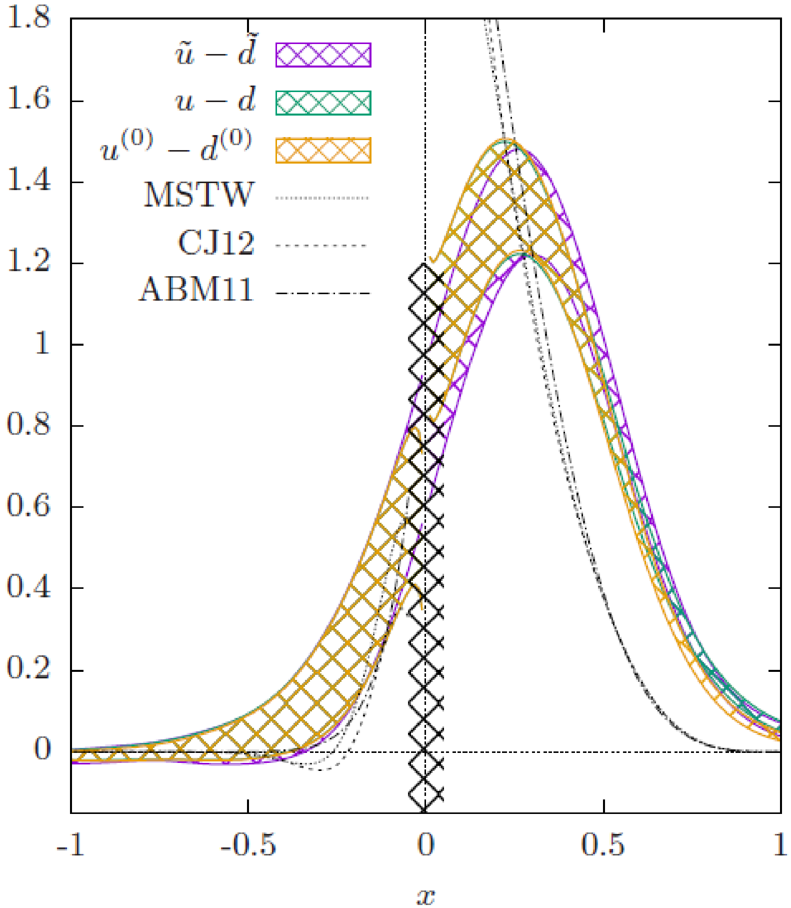
$$\boxed{\int_0^1 dx\left(\Delta\bar{u}(x) - \Delta\bar{d}(x)\right) = 0.184 \pm 0.047}$$

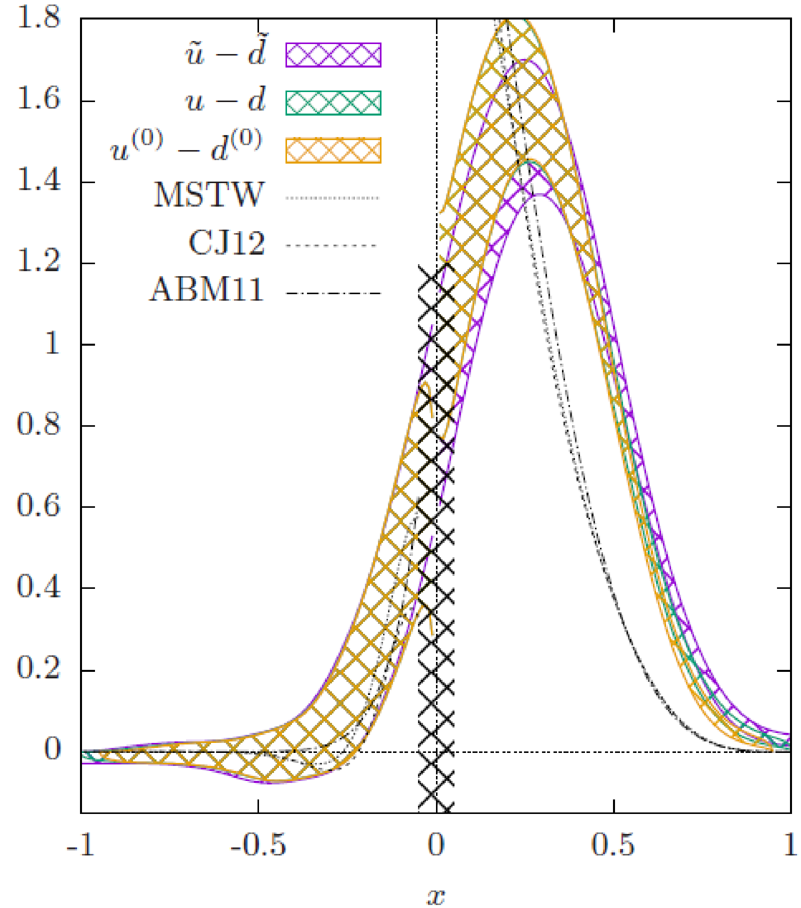
$$\boxed{\int_0^1 dx\left(\delta\bar{d}(x) - \delta\bar{u}(x)\right) = 0.169 \pm 0.047}$$

$$\Delta\bar{q}(x) = \Delta q(-x)$$

$$\delta\bar{q}(x) = -\delta q(-x)$$

$u(x) - d(x)$





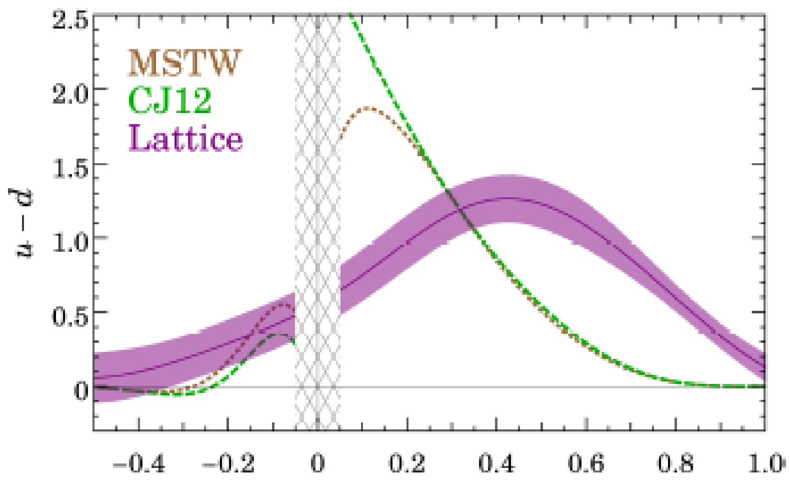
$P_3 = \frac{8\pi}{L}$ (1.96 GeV)

$P_3 = \frac{10\pi}{L}$ (2.45 GeV)

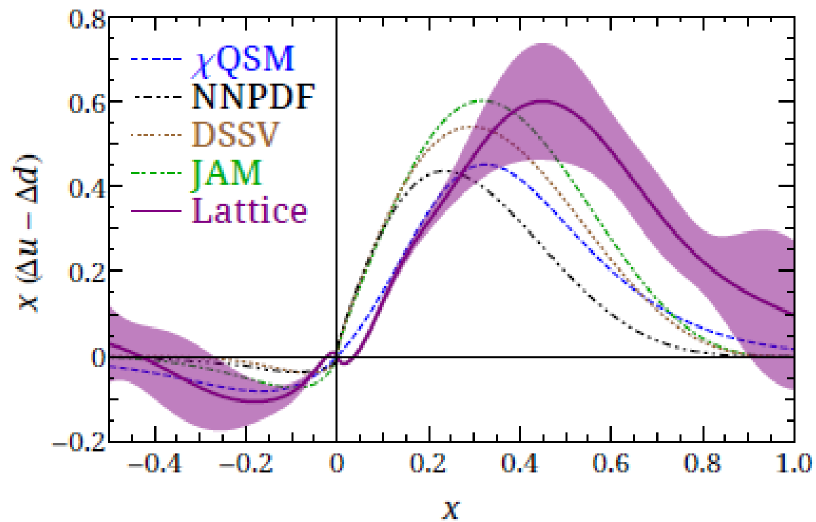
$$\int_0^1 dx\left(\bar{d}(x) - \bar{u}(x)\right) = 0.130 \pm 0.077$$

$$\int_0^1 dx\left(\bar{d}(x) - \bar{u}(x)\right) = 0.100 \pm 0.088$$

Only other result



Huey-Wen Lin et al., Phys. Rev. D91 (2015) 054510



J.-W. Chen et al., arXiv:1603.066664

$24^3 \times 48$

$a \approx 0.12fm \qquad N_f = 2+1+1$

$m_{PS} \approx 310\ MeV$

Uses highly improved staggered quarks and HYP smearing

Origin of the quark-antiquark asymmetry inside the proton

Matrix elements obeys the following relations:

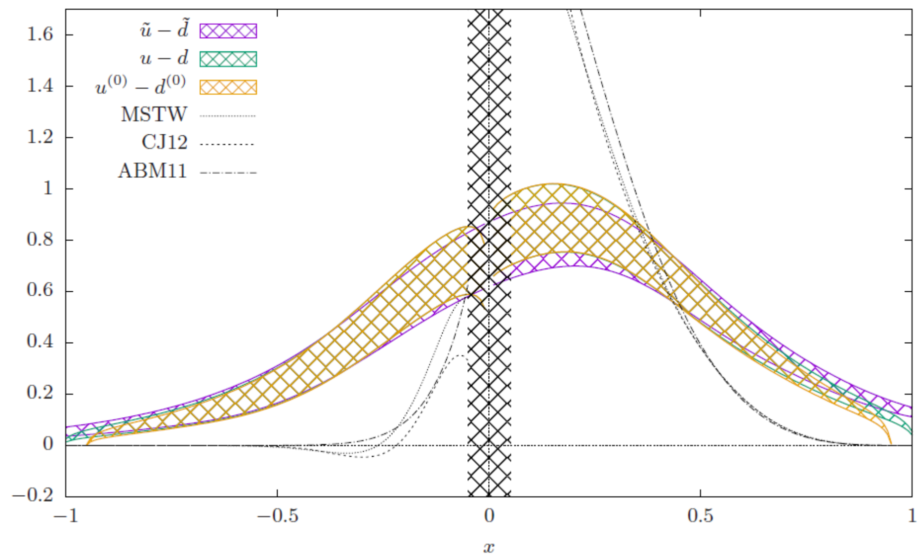
$$h(P_3, z) = h(P_3, -z)^\dagger$$

$$\Delta h(P_3, z) = \Delta h(P_3, -z)^\dagger$$

$$\delta h(P_3, z) = \delta h(P_3, -z)^\dagger$$

Imaginary part is odd under $z \to -z$

The asymmetry between x and $-x$ only appear because the imaginary part is an odd function



No HYP smearing in the gluon fields!!!

Renormalization seems to be fundamental for the asymmetry

Combined effect

Summary & Outline

- First attempts of a direct QCD calculation of quark distributions;
- Valuable information from intermediate to large x region;
- Asymmetric sea appears naturally. Imaginary part plays a fundamental role;
- Non perturbative renormalization is on its way;
- Momentum Smearing: it allows access to higher momentum;
- Compute at the physical mass – smaller number of configurations available at the moment;
- Go to the continuum;
- Singlet distributions, gluon distributions, TMDs, etc.
- Much to be done!

The Wilson twisted mass fermion action for the 2 light (u, d quarks) is given in the so-called twisted basis by: [R. Frezzotti, P. Grassi, G.C. Rossi, S. Sint, P. Weisz, 2000-2004]

$$S_l[\psi, \bar{\psi}, U] = a^4 \sum_x \bar{\chi}_l(x) \big(D_W + m_{0,l} + i\mu_l \gamma_5 \tau_3 \big) \chi_l(x),$$

where:

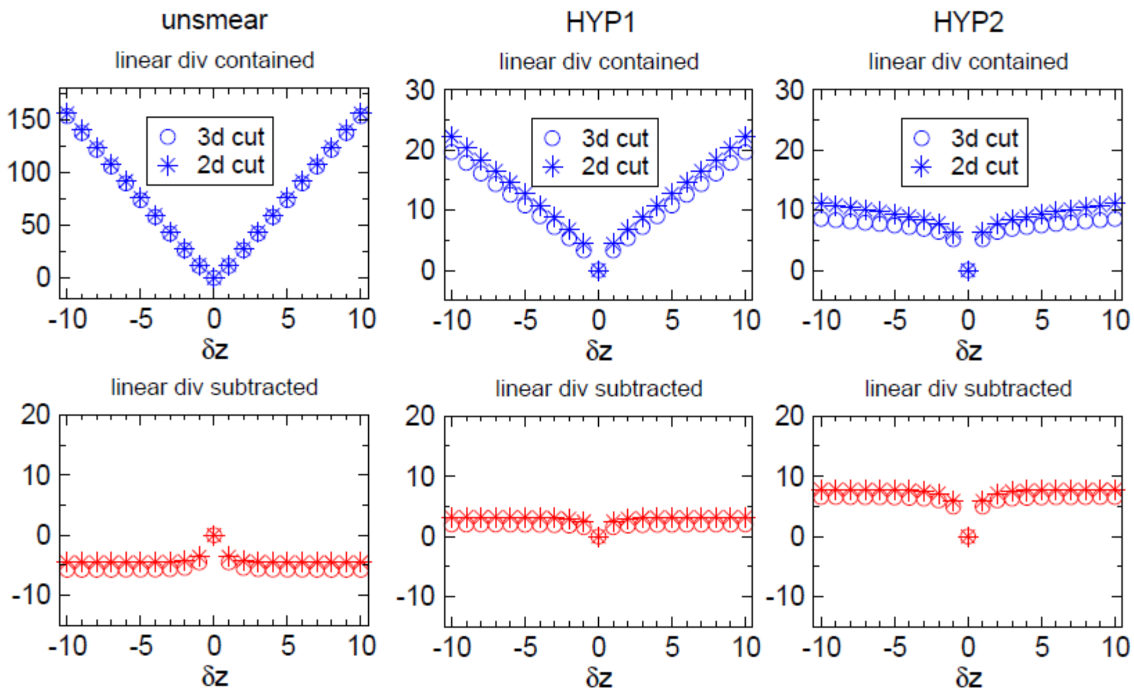
- D_W – Wilson-Dirac operator,
- $m_{0,l}$ and μ_l – bare untwisted and twisted light quark masses,
- the matrix τ^3 acts in flavour space,
- $\chi_l = (\chi_u, \chi_d)$ is a 2-component vector in flavour space, related to the one in the physical basis by a chiral rotation with angle ω:

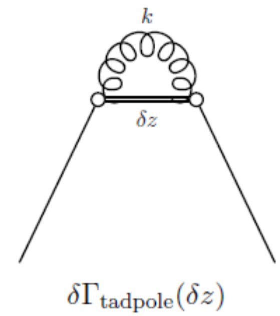
$$\psi = e^{i\gamma_5 \tau_3 \omega/2} \chi.$$

With maximal twist, $\omega = \pi/2$, automatic $\mathrm{O}(a)$-improvement is achieved.

Linear divergence and HYP smearing

T. Ishikawa, Y. Q. Ma, J. W. Qiu and S. Yoshida,
``Practical quasi parton distribution functions,"
arXiv:1609.02018.



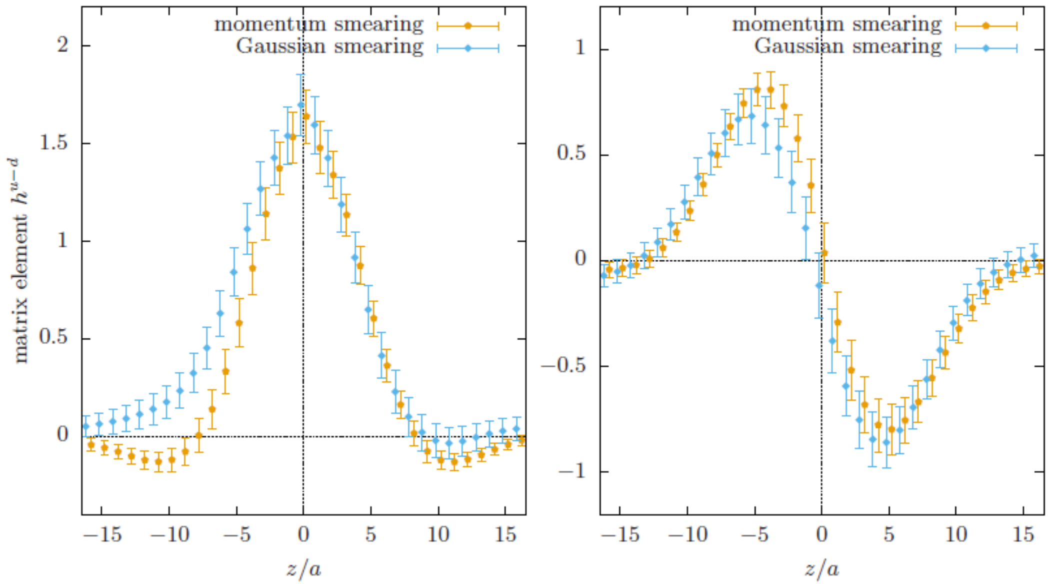


Linear divergences from tadpole-type diagrams

HYP smearing seems to eliminate the linear divergence from the lattice data

Gaussian and Momentum Smearing

- 30000 measurements for the case of Gaussian smearing;
- 150 measurements for the case of momentum smearing;
- We can now access larger values for the nucleon momentum;
- 150 measurements for the cases of $P_3 = \frac{6\pi}{L}, \frac{8\pi}{L}$;
- 300 measurements for the case of $P_3 = \frac{10\pi}{L}$.



$$P_3 = \frac{6\pi}{L}$$

$$\int_0^1 dx\left(\Delta\bar{u}(x)\ -\ \Delta\bar{d}(x)\right) = 0.184 \pm 0.047$$

$$\int_0^1 dx\left(\delta\bar{d}(x)\ -\ \delta\bar{u}(x)\right) = 0.169 \pm 0.047$$

$$\int_0^1 dx\left(\bar{d}(x)\ -\ \bar{u}(x)\right) = 0.187 \pm 0.055$$

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Mostrar o polarizado e o transversity somente para o caso de Momentum smearing